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A FINITE-DEFORMATION THEORY OF PLASTICITY

H. T. HAHN†
Nonmetallic Materials Division, Air Force Materials Laboratory,
Wright-Patterson AFB, Ohio 45433

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Abstract—Based on a multiplicative decomposition of local deformation into elastic and plastic deformations general constitutive equations of elastic–plastic materials are proposed. Two alternative approaches are discussed: one in which the elastic deformation is used as an independent variable, and the other in which the stress is one of the independent variables. The appropriate material symmetries are defined, and it is shown that the plastic spin is absent in the theory of isotropic materials. Analysis of a simple extension is given as an example.

1. INTRODUCTION

A typical stress–strain diagram of most metal bars in tension and compression tests reveals two distinct modes of deformation. Within a certain range of loading the stress–strain curve is reversible; the recoverable part may also include the thermal elongation caused by temperature changes. Outside this range a residual deformation remains after loading is removed. In classical plasticity this behavior is described by specifying constitutive equations for initial yielding, subsequent yielding, and plastic strain rate.[1]

A theory of plasticity which is closer to modern continuum mechanics of finite deformation begins with the work of Green and Naghdi[2]. They assume that the total Lagrangean strain $\mathbf{E}$ can be decomposed into elastic and plastic strains:

$$\mathbf{E} = \hat{\mathbf{E}} + \bar{\mathbf{E}}. \tag{1.1}$$

The plastic strain tensor $\bar{\mathbf{E}}$ was assumed symmetric at first, but later this assumption was abandoned[3]. The constitutive equations were similar to those in the classical theory except that the plastic strain rate was no longer assumed to be normal to the yield surface.

Alternatively, the deformation may be decomposed according to

$$\mathbf{C} = \hat{\mathbf{C}} + \bar{\mathbf{C}} \tag{1.2}$$

where $\mathbf{C}$ is the Cauchy–Green deformation tensor. This is done in the investigation by Perzyna and Wojno[4]. These authors used the work of Coleman and Gurtin[5] on thermodynamics for materials with internal state variables, which was also employed by Kratochvil and Dillon[6, 7]—together with the decomposition (1.1)—to set up theories of visco-plasticity. Constitutive equations for inviscid plastic materials were then obtained as limiting cases.

The multiplicative decomposition of the deformation gradients

$$\mathbf{F} = \hat{\mathbf{F}}\bar{\mathbf{F}}, \tag{1.3}$$

† NRC Postdoctoral Associate; formerly Graduate Assistant, Dept. of Engineering Mechanics, Pennsylvania State University, University Park, Penn. 16802

where $\hat{\mathbf{F}}$ and $\bar{\mathbf{F}}$ are the local elastic and plastic deformation tensors, respectively, has also been explored (cf. [8]). Specific formulations of plasticity theory have been proposed by Lee[9] and Freund[10], and also by Kelly[11] for visco-plastic materials.

In addition to these approaches, Green and Naghdi[12] showed that plasticity theory could be incorporated into Green and Rivlin's multipolar mechanics[13] by regarding the elastic deformation field as a dipolar displacement field. This correspondence was also pointed out by Fox[14].

A functional theory of rate-independent materials introduced by Pipkin and Rivlin[15] has further been developed by Owen[16, 17]. In particular, Owen showed how a plastic deformation $\bar{\mathbf{F}}$ could be obtained from special characteristics assumed for the functional for stress.

Although Green and Naghdi's work provides a general framework of the classical plasticity theory, the decomposition (1.3) seems more amenable to physical interpretation than the decomposition (1.1). Moreover, if the strains are defined in terms of $\hat{\mathbf{F}}$ and $\bar{\mathbf{F}}$, then the elastic strain rate $\dot{\hat{\mathbf{E}}}$ is easily seen to be affected by plastic deformation rate.

Theories based on (1.3) have so far been restricted to the case of isotropic materials under the assumption that the observer transformation can be applied as well to the plastic configuration [10, 17]. It will be shown in section 4, however, that this is so only for isotropic materials but not true otherwise.

In this paper, therefore, we formulate a general theory of plasticity based on the multiplicative decomposition (1.3), employing the approach of the classical plasticity theory. Requisite constitutive equations are proposed in section 2 regarding the plastic deformation $\bar{\mathbf{F}}$ as an internal state variable. Appropriate material symmetry is discussed in section 3. Two special cases of interest are then considered in the subsequent sections, followed by an example on a simple extension.

2. CONSTITUTIVE EQUATIONS

Before stating the constitutive equations for elastic-plastic materials, it is necessary to choose a set of variables suitable for describing plastic phenomena. This set includes a strain-hardening parameter κ that is monotonously increasing with plastic processes, and so expresses the irreversibility of plastic processes, or, in other words, plastic ageing of material† The basic thermomechanical variables—free energy density ψ, Cauchy stress $\mathbf{T}$, entropy density η, and heat flux $\mathbf{q}$—are then determined by the set of variables, $\{\mathbf{F}, \theta, \mathbf{g}, \bar{\mathbf{F}}, \kappa\}$, where θ is the temperature and $\mathbf{g}$ the temperature gradient. The changes of $\bar{\mathbf{F}}$ and κ are given by rate-type equations.

Introducing the ordered pairs

$$\mathbf{\Lambda} = (\mathbf{F}, \theta), \ \mathbf{\Pi} = (\bar{\mathbf{F}}, \kappa), \tag{2.1}$$

we set forth constitutive equations of an elastic–plastic material as follows,

$$\begin{gathered}\psi = \psi(\mathbf{\Lambda}, \mathbf{g}, \mathbf{\Pi}), \ \mathbf{T} = \mathbf{T}(\mathbf{\Lambda}, \mathbf{g}, \mathbf{\Pi}), \ \eta = \eta(\mathbf{\Lambda}, \mathbf{g}, \mathbf{\Pi}) \\ \mathbf{q} = \mathbf{q}(\mathbf{\Lambda}, \mathbf{g}, \mathbf{\Pi}), \dot{\mathbf{\Pi}} = \mathbf{H}[\mathring{\mathbf{\Lambda}}],\end{gathered} \tag{2.2}$$

where $\mathbf{H} = \mathbf{H}(\mathbf{\Lambda}, \mathbf{\Pi})$ is a linear operator on a ten-dimensional vector space, and

† The choice of κ is not unique. In fact, without conceptual difficulty, we could introduce a set of N parameters κ_i, $i = 1, 2, \ldots, N$, each of which representing, e.g. a dislocation density in a slip system in crystalline materials[18]. Since this extension is mathematically straightforward, however, we retain the single parameter κ only.

$$\mathring{\mathbf{\Lambda}} = (\mathring{\mathbf{F}}, \dot{\theta}), \mathring{\mathbf{F}} = \left.\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t}\right|_{\bar{\mathbf{F}}} = \dot{\mathbf{F}}\bar{\mathbf{F}} \tag{2.3}$$

Here a superposed dot denotes material time-derivatives.

Apart from (2.2) additional information must be given concerning the existence of an elastic range and a yield surface. Specifically, this is done by postulating a loading function Y such that the elastic range $E_F(\mathbf{\Pi})$ is described by

$$E_F(\mathbf{\Pi}) = \{\mathbf{\Lambda} \mid Y(\mathbf{\Lambda}, \mathbf{\Pi}) < 0\}, \tag{2.4}$$

and for all admissible processes the loading parameter Y is non-positive,

$$Y = Y(\mathbf{\Lambda}, \mathbf{\Pi}) \leq 0. \tag{2.5}$$

Accordingly, those processes passing through a point on a yield surface are divided into: (1) loading, i.e. $\dot{\mathbf{\Pi}} \neq \mathbf{0}$. $Y = 0$, $\mathring{Y} > 0$; (2) neutral loading, i.e. $\dot{\mathbf{\Pi}} = \mathbf{0}$ $Y = 0$, $\mathring{Y} = 0$; (3) unloading, i.e. $\dot{\mathbf{\Pi}} = \mathbf{0}$ $Y = 0$ $\mathring{Y} < 0$, where

$$\mathring{Y} = \mathrm{tr}(\partial_{\mathbf{\Lambda}} Y \mathring{\mathbf{\Lambda}}^T) = \mathrm{tr}(\partial_{\mathbf{F}} Y \mathring{\mathbf{F}}^T) + \partial_\theta Y \dot{\theta}. \tag{2.6}$$

Following Green and Naghdi[2] one can derive from the condition for neutral loading that

$$\mathbf{H} = (\lambda \mathbf{M} \otimes \partial_{\mathbf{F}} Y, \zeta \partial_\theta Y), \tag{2.7}$$

where $\otimes$ denotes a tensor product and

$$\lambda = \lambda(\mathbf{\Lambda}, \mathbf{\Pi}), \mathbf{M} = \mathbf{M}(\mathbf{\Lambda}, \mathbf{\Pi}), \zeta = \zeta(\mathbf{\Lambda}, \mathbf{\Pi}), \tag{2.8}$$

and without loss of generality λ is assumed positive. Therefore, the equation for $\dot{\mathbf{\Pi}}$ can be rewritten as

$$\dot{\mathbf{\Pi}} = \mathbf{\Gamma} \mathring{Y} H(Y) H(\mathring{Y}), \tag{2.9}$$

where $\mathbf{\Gamma}$ is the ordered pair

$$\mathbf{\Gamma} = (\lambda \mathbf{M}, \zeta), \tag{2.10}$$

and $H(\cdot)$ is the Heaviside step function.

The variables λ, $\mathbf{M}$ and ζ are not independent, because any loading process should satisfy

$$\dot{Y} = \mathring{Y} + \mathrm{tr}(\partial_{\mathbf{\Pi}} Y \dot{\mathbf{\Pi}}^T) + \mathrm{tr}[(\bar{\mathbf{F}}^{-1})^T \mathbf{F}^T \partial_{\mathbf{F}} Y \dot{\bar{\mathbf{F}}}^T] = 0. \tag{2.11}$$

In particular, from (2.9), (2.10) and (2.11) it follows that

$$1 + \mathrm{tr}(\partial_{\mathbf{\Pi}} Y \mathbf{\Gamma}^T) + \lambda \,\mathrm{tr}[(\bar{\mathbf{F}}^{-1})^T \mathbf{F}^T \partial_{\mathbf{F}} Y \mathbf{M}^T] = 0. \tag{2.12}$$

The constitutive equations are further restricted by the postulate of non-negative entropy production,

$$-\rho(\dot{\psi} + \eta\dot{\theta}) - \theta^{-1}\mathbf{q} \cdot \mathbf{g} + \mathrm{tr}(\mathbf{TL}) \geq 0, \tag{2.13}$$

$$\mathbf{L} \equiv \dot{\mathbf{F}}\mathbf{F}^{-1},$$

where ρ is the mass density. The above inequality is now applied to elastically admissible processes, in which $\dot{\mathbf{\Pi}} = \mathbf{0}$, to obtain

$$\psi = \psi(\mathbf{\Lambda}, \mathbf{\Pi}), (\rho^{-1}\mathbf{T}(\mathbf{F}^T)^{-1}, -\eta) = \partial_{\mathbf{\Lambda}} \psi(\mathbf{\Lambda}, \mathbf{\Pi}), \tag{2.14}$$

$$\mathbf{q} \cdot \mathbf{g} \leq 0. \tag{2.15}$$

In any process relations (2.14) remain valid if appropriate one-sided derivatives are used at those points on yield surfaces. However, the heat conduction inequality is replaced by the general dissipation inequality

$$\rho\theta\sigma - \theta^{-1}\mathbf{q}\cdot\mathbf{g} \geq 0, \tag{2.16}$$

where

$$\sigma = -\frac{1}{\theta}\operatorname{tr}(\partial_{\mathbf{\Pi}}\psi\dot{\mathbf{\Pi}}^T) = -\frac{1}{\theta}\operatorname{tr}(\partial_{\mathbf{\Pi}}\psi\mathbf{\Gamma}^T)\dot{Y}H(Y)H(\dot{Y}) \tag{2.17}$$

is the internal dissipation. Since σ is independent of $\mathbf{g}$, it further follows that

$$\operatorname{tr}(\partial_{\mathbf{\Pi}}\psi\mathbf{\Gamma}^T) \leq 0. \tag{2.18}$$

3. MATERIAL OBJECTIVITY AND MATERIAL SYMMETRY

According to the principle of material objectivity, constitutive equations should be invariant to all observer transformations. Therefore, $\mathbf{F}$ in the domain of $(2.2)_1$, (2.5) and (2.8) is replaced by the stretch $\mathbf{U}$, and

$$\mathbf{q} = \mathbf{R}\mathbf{q}(\mathbf{U}, \theta, \mathbf{R}^T\mathbf{g}, \mathbf{\Pi}), \tag{3.1}$$

where $\mathbf{R}$ is the orthogonal rotation tensor,

$$\mathbf{F} = \mathbf{R}\,\mathbf{U}. \tag{3.2}$$

Symmetry of an elastic–plastic material relative to a reference configuration is determined by the symmetry group of its response functions based on the chosen reference configuration. Given any response function f other than $\mathbf{M}$, the symmetry group S^f of f is the set of all unimodular tensors $\mathbf{H}$ that satisfy

$$f(\mathbf{F}, \theta, \mathbf{g}, \bar{\mathbf{F}}, \kappa) = f(\mathbf{F}\mathbf{H}, \theta, \mathbf{g}, \bar{\mathbf{F}}\mathbf{H}, \kappa) \tag{3.3}$$

for all $\mathbf{F}$, θ, $\mathbf{g}$, $\bar{\mathbf{F}}$, κ in the domain of f. For the function $\mathbf{M}$, the condition is modified as

$$\mathbf{M}(\mathbf{F}, \theta, \bar{\mathbf{F}}, \kappa)\mathbf{H} = \mathbf{M}(\mathbf{F}, \theta, \bar{\mathbf{F}}\mathbf{H}, \kappa) \tag{3.4}$$

for all $\mathbf{F}$, θ, $\bar{\mathbf{F}}$, κ in the domain of $\mathbf{M}$. If there exist reference configurations for which the symmetry group

$$S = S^{\psi} \cap S^{\mathbf{T}} \cap S^{\eta} \cap S^{\mathbf{q}} \cap S^{Y} \cap S^{\lambda} \cap S^{\mathbf{M}} \cap S^{\zeta} \tag{3.5}$$

contains the orthogonal group, then the material is called isotropic and these configurations are said to be undistorted.

At a fixed $\mathbf{\Pi} = (\bar{\mathbf{F}}, \kappa)$ an elastic–plastic material particle behaves like an elastic material. Therefore, it is also of interest to know any symmetry the particle may possess in that state. To this end we take the plastic configuration defined by $\bar{\mathbf{F}}$ as a reference configuration and write

$$f(\mathbf{F}, \theta, \mathbf{g}, \mathbf{\Pi}) = f_{\mathbf{\Pi}}(\mathbf{F}\bar{\mathbf{F}}^{-1}, \theta, \mathbf{g}) \tag{3.6}$$

for any constitutive equation. The instantaneous symmetry group at $\mathbf{\Pi}$, $S(\mathbf{\Pi})$, then consists of unimodular tensors $\mathbf{H}$ that satisfy

$$\begin{aligned} f_{\mathbf{\Pi}}(\hat{\mathbf{F}}, \theta, \mathbf{g}) &= f_{\mathbf{\Pi}}(\hat{\mathbf{F}}\mathbf{H}, \theta, \mathbf{g}), \\ \mathbf{H}^{-1}\mathbf{M}_{\mathbf{\Pi}}(\hat{\mathbf{F}}, \theta) &= \mathbf{M}_{\mathbf{\Pi}}(\hat{\mathbf{F}}\mathbf{H}, \theta) \end{aligned} \tag{3.7}$$

for all $\hat{\mathbf{F}}$, θ and $\mathbf{g}$. The motivation for this definition will become clear in the next section.

Thus the change of (instantaneous) isotropy with plastic deformation is described by the dependence of the instantaneous symmetry group on $(\overline{\mathbf{F}}, \kappa)$. An example of this phenomenon is the appearance of anisotropy after metals are cold rolled.

4. ELASTIC DEFORMATION TENSOR AS INDEPENDENT VARIABLE

For crystalline solids it is known that plastic flow is caused mainly by the motion of dislocations and that the number and arrangement of dislocations are the prime sources of plastic properties, e.g. strain-hardening[18]. Since the tensor $\overline{\mathbf{F}}$ denotes the resulting deformation from a reference configuration, it is plausible to assume that the response functions depend on $\mathbf{F}$ and $\overline{\mathbf{F}}$ through the elastic deformation $\hat{\mathbf{F}} = \mathbf{F}\overline{\mathbf{F}}^{-1}$, and the dependence on plastic deformation enters only through the strain-hardening parameter κ. To proceed (2.9) is modified as

$$\begin{aligned}\overline{\mathbf{L}} \equiv \dot{\overline{\mathbf{F}}}\overline{\mathbf{F}}^{-1} &= \lambda(\mathbf{F}, \theta, \overline{\mathbf{F}}, \kappa)\mathbf{M}(\mathbf{F}, \theta, \overline{\mathbf{F}}, \kappa)\overline{\mathbf{F}}^{-1}\,\dot{Y}H(Y)H(\dot{Y}) \\ &= \lambda(\mathbf{F}, \theta, \overline{\mathbf{F}}, \kappa)\mathbf{N}(\mathbf{F}, \theta, \overline{\mathbf{F}}, \kappa)\,\dot{Y}H(Y)H(\dot{Y}),\end{aligned} \tag{4.1}$$

where we have introduced a new function $\mathbf{N}$,

$$\mathbf{N}(\mathbf{F}, \theta, \overline{\mathbf{F}}, \kappa) \equiv \mathbf{M}(\mathbf{F}, \theta, \overline{\mathbf{F}}, \kappa)\overline{\mathbf{F}}^{-1}. \tag{4.2}$$

The simplified equations are then

$$\psi = \hat{\psi}(\hat{\mathbf{U}}, \theta, \kappa),\ \mathbf{T} = \rho\hat{\mathbf{R}}\,\partial_{\hat{\mathbf{U}}}\hat{\psi}\,\hat{\mathbf{U}}^T\hat{\mathbf{R}}^T,\ \eta = -\partial_\theta\hat{\psi}, \tag{4.3}$$

$\mathbf{q} = \hat{\mathbf{R}}\mathbf{q}(\hat{\mathbf{U}}, \theta, \hat{\mathbf{R}}^T\mathbf{g}, \kappa)$, $Y = \hat{Y}(\hat{\mathbf{U}}, \theta, \kappa)$, and so on. Here the material objectivity has been used to obtain the reduced forms. Furthermore, (2.6) reduces to

$$\dot{Y} = \mathrm{tr}(\partial_{\hat{\mathbf{U}}}\hat{Y}\hat{\mathbf{U}}) + \partial_\theta\hat{Y}\dot{\theta}. \tag{4.4}$$

Definition of a symmetry group $S(\kappa)$ at κ is entirely analogous to (3.7) except that for $\hat{\mathbf{N}}$ a symmetry operation $\mathbf{H}$ satisfies

$$\mathbf{H}^{-1}\hat{\mathbf{N}}(\hat{\mathbf{F}}, \theta, \kappa)\mathbf{H} = \hat{\mathbf{N}}(\hat{\mathbf{F}}\mathbf{H}, \theta, \kappa) \tag{4.5}$$

identically for $\hat{\mathbf{F}}$ and θ. Moreover, $(4.3)_2$ and $(4.3)_3$ can be used to show that $S^{\psi}(\kappa)$ is a subgroup of both $S^{\hat{\mathbf{T}}}(\kappa)$ and $S^{\hat{\eta}}(\kappa)$[19].

Any material defined by the constitutive equations (4.3) is insensitive to the change of reference configurations. Consequently it is not expedient to use (3.3)–(3.5) to define its symmetry. Therefore, we redefine the isotropy of an elastic–plastic material described by (4.3) as follows: An elastic–plastic material is called isotropic at κ if $S(\kappa)$ contains the orthogonal group; moreover, if the material is isotropic at every κ, then it is said to be isotropic. Hereafter, isotropy will be understood in this sense.

If the material is isotropic at κ, i.e., all the response functions at κ are form-invariant for all orthogonal transformations, the plastic spin at κ is absent. To show this we first note that $\overline{\mathbf{L}} = \overline{\mathbf{D}} + \overline{\mathbf{W}}$, where $\overline{\mathbf{D}}$ is the plastic stretching and $\overline{\mathbf{W}}$ the plastic spin. Therefore, $\hat{\mathbf{N}}$ will consist of two functions $\hat{\mathbf{N}}_S$ and $\hat{\mathbf{N}}_W$ such that

$$\begin{aligned}\overline{\mathbf{D}} &= \lambda\hat{\mathbf{N}}_S(\hat{\mathbf{U}}, \theta, \kappa)\,\dot{Y}H(Y)H(\dot{Y}), \\ \overline{\mathbf{W}} &= \lambda\hat{\mathbf{N}}_W(\hat{\mathbf{U}}, \theta, \kappa)\,\dot{Y}H(Y)H(\dot{Y}).\end{aligned} \tag{4.6}$$

Since $\overline{\mathbf{W}}$ is skew-symmetric and $\hat{\mathbf{U}}$ is symmetric, we can easily extend Wang's representation theorem[20] to show that (cf. 5.8).

$$\overline{\mathbf{W}} = 0. \tag{4.7}$$

for an isotropic material body.

A physical explanation for this result is given as follows. An isotropic material particle also has isotropic distribution of dislocations. Therefore, there will be the same amount of dislocation flow on any two perpendicular planes toward the line of intersection, thus resulting in no plastic spin (cf.[18]).

The reduced forms of response functions and the absence of the plastic spin can also be derived from the assumption that the observer transformation applies as well to the plastic configuration, as was done in [10] and [17]. However, it should be noted again that this reduction is valid only for isotropic materials in the present theory.

We now turn our attention to the internal dissipation inequality (2.18). Since

$$\rho \operatorname{tr}(\partial_{\bar{\mathbf{F}}} \psi \dot{\bar{\mathbf{F}}}^T) = \rho \operatorname{tr}[\partial_{\hat{\mathbf{F}}} \hat{\psi} (\mathbf{F} \dot{\overline{\bar{\mathbf{F}}^{-1}}})^T] = -\rho \operatorname{tr}(\hat{\mathbf{F}}^{-1} \partial_{\hat{\mathbf{F}}} \psi \hat{\mathbf{F}}^T \bar{\mathbf{F}} \bar{\mathbf{L}}) = -J^{-1} \operatorname{tr}(\mathbf{P} \bar{\mathbf{L}}), \tag{4.8}$$

where

$$J \equiv \det \mathbf{F}, \ \mathbf{P} \equiv J \hat{\mathbf{F}}^{-1} \mathbf{T} \hat{\mathbf{F}}, \tag{4.9}$$

the internal dissipation σ becomes

$$\sigma = \frac{\lambda}{\rho \theta} \left[\frac{1}{J} \operatorname{tr}(\mathbf{P} \mathbf{N}) - \rho \, \partial_\kappa \hat{\psi} \, \frac{\zeta}{\lambda} \right] \dot{\bar{Y}}. \tag{4.10}$$

Noting further that

$$\lambda \dot{\bar{Y}} = \operatorname{tr}(\mathbf{N}^T \bar{\mathbf{L}}) / \operatorname{tr}(\mathbf{N} \mathbf{N}^T),$$

we can write (4.10) as

$$\sigma = \frac{1}{\rho_0 \, \theta} \operatorname{tr}[(\mathbf{P} - \mathbf{P}_R) \bar{\mathbf{L}}] \geq 0. \tag{4.11}$$

where $\mathbf{P}_R$ is defined by

$$\mathbf{P}_R = \frac{\rho_0 \zeta}{\lambda} \partial_\kappa \hat{\psi} \, \frac{\mathbf{N}^T}{\operatorname{tr}(\mathbf{N} \mathbf{N}^T)} \tag{4.12}$$

and ρ_0 is the initial mass density, $\rho_0 = J\rho$.

As is clear from its definition, $\mathbf{P}_R$ has the dimension of stress and accounts for the change of the free energy due to plastic flow, i.e. the change of energy associated with dislocations.

It is also interesting to observe that the possibility of negative plastic working,

$$\operatorname{tr}(\mathbf{P} \bar{\mathbf{L}}) < 0,$$

is not ruled out. An experimental evidence and discussion concerning this may be found in[21]. The above inequality also points out the shortcomings of defining the plastic deformation through a relaxation process.

5. ELASTIC RANGE IN STRESS SPACE

Frequently plastic behavior is described by constitutive equations the domain of which includes a stress space instead of a deformation space, thus

$$Y = \bar{Y}(\mathbf{P}, \theta, \kappa), \ \lambda = \bar{\lambda}(\mathbf{P}, \theta, \kappa), \ \mathbf{N} = \bar{\mathbf{N}}(\mathbf{P}, \theta, \kappa), \ \zeta = \bar{\zeta}(\mathbf{P}, \theta, \kappa), \tag{5.1}$$

and $\dot{Y}$ is given by

$$\dot{Y} = \mathrm{tr}(\partial_{\mathbf{P}} \overline{Y} \dot{\mathbf{P}}^T) + \partial_\theta \overline{Y} \dot{\theta}, \quad \dot{\mathbf{P}} = \mathrm{tr}(\partial_{\hat{\mathbf{F}}} \mathbf{P} \dot{\hat{\mathbf{F}}}^T) + \partial_\theta \mathbf{P} \dot{\theta}. \tag{5.2}$$

At a fixed κ, elastic behavior of an elastic–plastic material is described by "elastic" constitutive equations $(4\text{–}3)_{1-4}$, and plastic behavior by "plastic" constitutive equations (5.1). Therefore, it seems plausible to define the elastic symmetry group at κ, $\hat{S}(\kappa)$, and the plastic group at κ, $\bar{S}(\kappa)$, respectively, by

$$\begin{aligned} \hat{S}(\kappa) &= S^{\hat{\psi}}(\kappa) \cap S^{\hat{\mathbf{T}}}(\kappa) \cap S^{\hat{\eta}}(\kappa) \cap S^{\hat{\mathbf{q}}}(\kappa), \\ \bar{S}(\kappa) &= S^{\overline{Y}}(\kappa) \cap S^{\bar{\lambda}}(\kappa) \cap S^{\bar{\mathbf{N}}}(\kappa) \cap S^{\bar{\xi}}(\kappa), \end{aligned} \tag{5.3}$$

with the understanding that an element of $\mathbf{H}$ of $\bar{S}(\kappa)$ satisfies

$$\begin{aligned} f(\mathbf{P}, \theta, \kappa) &= f(\mathbf{H}^{-1}\mathbf{P}\mathbf{H}, \theta, \kappa), \\ \mathbf{H}^{-1}\overline{\mathbf{N}}(\mathbf{P}, \theta, \kappa)\mathbf{H} &= \overline{\mathbf{N}}(\mathbf{H}^{-1}\mathbf{P}\mathbf{H}, \theta, \kappa) \end{aligned} \tag{5.4}$$

identically for $\mathbf{P}$ and θ. Here f is any one of scalar functions $\overline{Y}$, $\bar{\lambda}$ and $\bar{\xi}$. It is easily seen that $S(\kappa)$ of the previous section is a subgroup of $\hat{S}(\kappa)$. Correspondingly, an elastic–plastic material is said to be elastically isotropic at κ if $\hat{S}(\kappa)$ is the orthogonal group and plastically isotropic at κ if $\bar{S}(\kappa)$ is the orthogonal group.

Apart from this symmetry operation, the loading function $\overline{Y}$ may become invariant to the transformation $\mathbf{P} \to -\mathbf{P}$. When this happens, the material is said to have no Bauschinger effect.

The stress $\mathbf{P}$ does not depend on $\overline{\mathbf{F}}$ because

$$\mathbf{P} = \rho_0 \hat{\mathbf{U}}^{-1} \partial_{\hat{U}} \hat{\psi} \hat{\mathbf{U}}^T \hat{\mathbf{U}} = \hat{\mathbf{P}}(\hat{\mathbf{U}}, \theta, \kappa). \tag{5.5}$$

Thus the elastic range $E_F(\kappa)$ in deformation–temperature space is related to the elastic range $E_P(\kappa)$ in stress–temperature space through

$$\overline{Y}(\mathbf{P}, \theta, \kappa) = \overline{Y}(\hat{\mathbf{P}}(\hat{\mathbf{U}}, \theta, \kappa), \theta, \kappa) = \hat{Y}(\hat{\mathbf{U}}, \theta, \kappa). \tag{5.6}$$

From this also follows a relation between the normals:

$$(\partial_{\hat{U}_{ij}} \hat{Y}, \partial_\theta \hat{Y}) = (\partial_{\hat{U}_{ij}} \hat{P}_{km}\, \partial_{P_{km}} \overline{Y}, \partial_\theta \hat{P}_{km}\, \partial_{P_{km}} \overline{Y} + \partial_\theta \overline{Y}). \tag{5.7}$$

If the material is plastically isotropic, we obtain the following representations for the response functions:[20]†

$$\begin{aligned} Y &= \overline{Y}(I, \theta, \kappa) \\ \lambda \mathbf{N}_S &= d_0 \mathbf{1} + d_1 \mathbf{P}_S + d_2 \mathbf{P}_S{}^2 + d_3 \mathbf{P}_W{}^2 + d_4(\mathbf{P}_S \mathbf{P}_W - \mathbf{P}_W \mathbf{P}_S) \\ &\quad + d_5 \mathbf{P}_W \mathbf{P}_S \mathbf{P}_W + d_6(\mathbf{P}_S{}^2 \mathbf{P}_W - \mathbf{P}_W \mathbf{P}_S{}^2) + d_7 \mathbf{P}_W \mathbf{P}_S{}^2 \mathbf{P}_W \\ &\quad + d_8(\mathbf{P}_W \mathbf{P}_S \mathbf{P}_W{}^2 - \mathbf{P}_W{}^2 \mathbf{P}_S \mathbf{P}_W), \\ \lambda \mathbf{N}_W &= e_1 \mathbf{P}_W + e_2(\mathbf{P}_S \mathbf{P}_W + \mathbf{P}_W \mathbf{P}_S) + e_3(\mathbf{P}_S \mathbf{P}_W{}^2 - \mathbf{P}_W{}^2 \mathbf{P}_S), \\ \zeta &= \bar{\zeta}(I, \theta, \kappa), \end{aligned} \tag{5.8}$$

where I is the set of invariants,

$$I = \{\mathrm{tr}\, \mathbf{P}_S, \mathrm{tr}\, \mathbf{P}_S{}^2, \mathrm{tr}\, \mathbf{P}_S{}^3, \mathrm{tr}\, \mathbf{P}_W{}^2, \mathrm{tr}(\mathbf{P}_S \mathbf{P}_W{}^2), \mathrm{tr}(\mathbf{P}_S{}^2 \mathbf{P}_W{}^2), \mathrm{tr}(\mathbf{P}_S{}^2 \mathbf{P}_W{}^2 \mathbf{P}_S \mathbf{P}_W)\},$$

† The subscripts S and W denote the symmetric and skew-symmetric parts, respectively.

and d_i's and e_i's are scalar-valued functions of I, θ and κ. If the material is also elastically isotropic, then the stress $\mathbf{P}$ becomes symmetric,

$$\frac{1}{\rho_0}\mathbf{P} = \hat{\mathbf{U}}^{-1}\partial_{\hat{\mathbf{U}}}\hat{\psi}\hat{\mathbf{U}}^T\hat{\mathbf{U}} = \partial_{\hat{\mathbf{U}}}\hat{\psi}(\hat{\mathbf{U}}, \theta, \kappa)\hat{\mathbf{U}}^T, \tag{5.9}$$

and so the plastic spin $\overline{\mathbf{W}} = \lambda\overline{\mathbf{N}}_W\dot{Y}$ vanishes (cf. 4.7).

When yielding is independent of hydrostatic pressure, the loading function $\overline{Y}$ should satisfy

$$\overline{Y}(\mathbf{P}, \theta, \kappa) = \overline{Y}(\mathbf{P} + \mathbf{1}p, \theta, \kappa) \tag{5.10}$$

for any p because

$$\operatorname{tr}\mathbf{P} = J\operatorname{tr}(\hat{\mathbf{F}}^{-1}\mathbf{T}\hat{\mathbf{F}}) = J\operatorname{tr}\mathbf{T}. \tag{5.11}$$

In particular, taking

$$p = -\tfrac{1}{3}\operatorname{tr}\mathbf{P} \tag{5.12}$$

results in

$$\overline{Y}(\mathbf{P}, \theta, \kappa) = \overline{Y}(\mathbf{P}', \theta, \kappa), \tag{5.13}$$

where $\mathbf{P}'$ is the deviatoric part of $\mathbf{P}$. The elastic range $E_p(\kappa)$ is now a region in a nine-dimensional $\mathbf{P}'$–θ space.

Another special case of interest is that of a rigid-plastic material, for which elastic deformation is always a rotation,

$$\hat{\mathbf{F}} = \hat{\mathbf{R}}, \hat{\mathbf{U}} = \mathbf{1}. \tag{5.14}$$

Since

$$\mathbf{P} = J\hat{\mathbf{F}}^{-1}\mathbf{T}\hat{\mathbf{F}} = \bar{J}\hat{\mathbf{R}}^T\mathbf{T}\hat{\mathbf{R}}, \bar{J} \equiv \det\overline{\mathbf{F}}, \tag{5.15}$$

(5.2) is replaced by

$$\begin{aligned}
&\dot{Y} = \operatorname{tr}(\partial_{\mathbf{P}}\overline{Y}\dot{\mathbf{P}}^T) + \partial_\theta\overline{Y}\dot{\theta},\\
&\dot{\mathbf{P}} = \bar{J}\hat{\mathbf{R}}^T(\mathbf{T}\operatorname{tr}\overline{\mathbf{D}} - \hat{\mathbf{W}}\mathbf{T} + \mathbf{T}\hat{\mathbf{W}} + \dot{\mathbf{T}})\hat{\mathbf{R}},\\
&\hat{\mathbf{W}} \equiv \tfrac{1}{2}(\hat{\mathbf{L}} - \hat{\mathbf{L}}^T), \hat{\mathbf{L}} \equiv \dot{\hat{\mathbf{F}}}\hat{\mathbf{F}}^{-1}.
\end{aligned} \tag{5.16}$$

In addition, if the material is isotropic, the response functions (5.1) reduce to

$$Y = \overline{Y}(\bar{J}\mathbf{T}, \theta, \kappa), \lambda = \bar{\lambda}(\bar{J}\mathbf{T}, \theta, \kappa), \zeta = \bar{\zeta}(\bar{J}\mathbf{T}, \theta, \kappa), \mathbf{N}_S = \hat{\mathbf{R}}^T\overline{\mathbf{N}}(\bar{J}\mathbf{T}, \theta, \kappa)\hat{\mathbf{R}}, \mathbf{N}_W = \mathbf{0}. \tag{5.17}$$

6. AN EXAMPLE

As an illustration we consider the simple extension of an isotropic, plastically incompressible elastic–plastic solid without including thermal effects. In particular, constitutive equations are taken to be

$$\begin{aligned}
&\mathbf{T} = c_0\mathbf{1} + c_1\hat{\mathbf{B}} + c_2\hat{\mathbf{B}}^{-1}, \hat{\mathbf{B}} = \hat{\mathbf{F}}\hat{\mathbf{F}}^T,\\
&Y = S - \kappa, S = \tfrac{1}{2}\operatorname{tr}(\mathbf{T}'\mathbf{T}'^T), \mathbf{T}' = \mathbf{T} - \tfrac{1}{3}\mathbf{1}\operatorname{tr}\mathbf{T},\\
&\overline{\mathbf{L}} = \lambda\mathbf{N}\dot{Y}H(Y)\,H(\dot{Y}) = \frac{\lambda\mathbf{T}'}{\sqrt{(2S)}}\dot{S}H(Y)H(\dot{S}),\\
&\dot{\kappa} = \dot{S}H(Y)H(\dot{S}), \kappa_0 = \text{initial value of } \kappa,
\end{aligned} \tag{6.1}$$

where c_0, c_1, c_2 are scalar functions of the principal invariants $I_{\mathbf{B}}$, $II_{\mathbf{B}}$, $III_{\mathbf{B}}$ and λ depends on S which is one half the second principal moment of $\mathbf{T}'$.

A simple extension is described by

$$x = \alpha U X,\ Y = \alpha U Y,\ z = UZ. \tag{6.2}$$

where U is the stretch in the z direction and α denotes the lateral contraction. The tensors $\mathbf{F}$ and $\mathbf{B}$ are then

$$\mathbf{F} = \begin{bmatrix} \alpha U & 0 & 0 \\ 0 & \alpha U & 0 \\ 0 & 0 & U \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \alpha^2 U^2 & 0 & 0 \\ 0 & \alpha^2 U^2 & 0 \\ 0 & 0 & U^2 \end{bmatrix}, \tag{6.3}$$

respectively.

(*a*) *Elastic deformation before yielding*

Because $\mathbf{F} = \hat{\mathbf{F}}$, the Cauchy stresses are

$$\begin{aligned} T_{xx} &= T_{yy} = c_0 + c_1 \alpha^2 U^2 + c_2 \alpha^{-2} U^{-2}, \\ T_{zz} &= c_0 + c_1 U^2 + c_2 U^{-2}. \end{aligned} \tag{6.4}$$

The undetermined coefficient α is chosen so as to clear the lateral sides from the tractions T_{xx} and T_{yy}:

$$c_0 + c_1 \alpha^2 U^2 + c_2 \alpha^{-2} U^{-2} = 0. \tag{6.5}$$

The stretch U is restricted to those satisfying

$$\hat{Y}(U, \kappa_0) = S(U) - \kappa_0 < 0. \tag{6.6}$$

(*b*) *Incipience of yielding*

Yielding occurs when the stretch U reaches U_0 which is determined by

$$\hat{Y}(U_0, \kappa_0) = 0. \tag{6.7}$$

(*c*) *Elastic–Plastic deformation*

Because the material is isotropic, $\hat{\mathbf{F}}$ and $\overline{\mathbf{F}}$ will be of the forms

$$\hat{\mathbf{F}} = \begin{bmatrix} \hat{\alpha}\hat{U} & 0 & 0 \\ 0 & \hat{\alpha}\hat{U} & 0 \\ 0 & 0 & \hat{U} \end{bmatrix} \qquad \overline{\mathbf{F}} = \begin{bmatrix} \bar{U} & 0 & 0 \\ 0 & \bar{U} & 0 \\ 0 & 0 & \bar{U} \end{bmatrix}, U > U_0, \tag{6.8}$$

respectively. Hence the plastic deformation rate $\overline{\mathbf{L}}$ becomes

$$\overline{\mathbf{L}} = \begin{bmatrix} \dot{\bar{U}}/\bar{U} & 0 & 0 \\ 0 & \dot{\bar{U}}/\bar{U} & 0 \\ 0 & 0 & \dot{\bar{U}}/\bar{U} \end{bmatrix}. \tag{6.9}$$

The Cauchy stress $\mathbf{T}$ is determined from (6.4) with U and α replaced by $\hat{U}$ and $\hat{\alpha}$, respectively, whence the deviatoric stresses are given by

$$T_{xx}' = T_{yy}' = c_1 \hat{U}^2 \frac{\hat{\alpha}^2 - 1}{3} + c_2 \hat{U}^{-2} \frac{\hat{\alpha}^{-2} - 1}{3}, \tag{6.10}$$

$$T_{zz}' = c_1 \hat{U}^2 \frac{2(1 - \hat{\alpha}^2)}{3} + c_2 \hat{U}^{-2} \frac{2(1 - \hat{\alpha}^{-2})}{3},$$

where $\hat{\alpha}$ should satisfy (cf. 6.5)

$$c_0 + c_1\hat{\alpha}^2\hat{U}^2 + c_2\hat{\alpha}^{-2}\hat{U}^{-2} = 0.$$

The constitutive equation $(6.1)_6$ for $\bar{\mathbf{L}}$ now becomes

$$\bar{\mathbf{L}}_{xx} = \bar{\mathbf{L}}_{yy} = \frac{\dot{\bar{U}}}{\bar{U}} = \frac{\lambda T_{xx}'}{\sqrt{(2S)}}\frac{\mathrm{d}S}{\mathrm{d}\hat{U}}\dot{\hat{U}},$$

$$\bar{\mathbf{L}}_{zz} = \frac{\dot{\bar{U}}}{\bar{U}} = \frac{\lambda T_{zz}'}{\sqrt{(2S)}}\frac{\mathrm{d}S}{\mathrm{d}\hat{U}}\dot{\hat{U}}. \tag{6.11}$$

Consequently, the relation between $\hat{\mathbf{F}}$ and $\bar{\mathbf{F}}$ is obtained by integrating the above equations:

$$\ln \tilde{U} = \int_{U_0}^{\hat{U}} \frac{\lambda T_{xx}'}{\sqrt{(2S)}}\frac{\mathrm{d}S}{\mathrm{d}\hat{U}}\,\mathrm{d}\hat{U}$$

$$\ln \bar{U} = \int_{U_0}^{\hat{U}} \frac{\lambda T_{zz}'}{\sqrt{(2S)}}\frac{\mathrm{d}S}{\mathrm{d}\hat{U}}\,\mathrm{d}\hat{U}, \tag{6.12}$$

or, equivalently,

$$\tilde{U} = \tilde{U}(\hat{U}, U_0) = \exp\left(\int_{U_0}^{\hat{U}} \frac{\lambda T_{xx}'}{\sqrt{(2S)}}\frac{\mathrm{d}S}{\mathrm{d}\hat{U}}\,\mathrm{d}\hat{U}\right),$$

$$\bar{U} = \bar{U}(\hat{U}, U_0) = \exp\left(\int_{U_0}^{\hat{U}} \frac{\lambda T_{zz}'}{\sqrt{(2S)}}\frac{\mathrm{d}S}{\mathrm{d}\hat{U}}\,\mathrm{d}\hat{U}\right). \tag{6.13}$$

Determination of the elastic stretch $\hat{U}$ in terms of the total stretch U may be possible with the aid of the compatibility condition $\mathbf{F} = \hat{\mathbf{F}}\bar{\mathbf{F}}$. That is,

$$U = \hat{U}\bar{U} = \hat{U}\bar{U}(\hat{U}, U_0),$$

and further, the coefficient α is determined from

$$\alpha = \frac{\hat{\alpha}\hat{U}\tilde{U}}{U} = \frac{\hat{\alpha}\tilde{U}(\hat{U}, U_0)}{\bar{U}(\hat{U}, U_0)}. \tag{6.14}$$

Thus, given a deformation of the form (6.2), the corresponding elastic deformation (or stress) and plastic deformation can be determined.

7. CONCLUDING REMARKS

So far theories based on a multiplicative decomposition of local deformation into elastic and plastic deformations were by and large developed for isotropic materials only. In this paper we have proposed a general theory without this restriction. This theory was then simplified in the light of the observation that elastic properties are essentially preserved when deformation is measured from a plastic configuration. Since it is a usual practice in plasticity to use stress as an independent variable, this possibility was also explored in connection with the other formulations. Material symmetries were discussed, which show the change of instantaneous isotropy with plastic deformation. It is an immediate result of the proposed theory that isotropic elastic–plastic materials have no plastic spin.

The present theory can easily be extended to formulate a theory of elastic–viscoplastic materials. To this end we simply replace $\lambda\dot{Y}$ and $\zeta\dot{Y}$ by λ_v and ζ_v, respectively. The loading

parameter Y is now any real number, not necessarily non-positive (cf. 2.5). The parameter λ_v is a measure of the magnitude of plastic flow and the parameter ζ_v represents the time-rate of change of the strain hardening parameter κ. They are functions of $\mathbf{F}$, θ, $\overline{\mathbf{F}}$ and κ.

The theory of section 4 can also describe perfectly plastic behavior with minor modifications. Since the yield function $\hat{Y}$ is independent of κ, we have during a loading process

$$\mathring{\hat{Y}} = \dot{Y} = 0,$$

and so λ and ζ tends to infinity so that $\lambda \mathring{\hat{Y}} = \lambda_0$ and $\zeta \mathring{\hat{Y}} = \zeta_0$ remain finite. The constitutive equations for λ_0 and ζ_0 are given by

$$\lambda_0 = \hat{\lambda}_0(\hat{\mathbf{U}}, \theta, \kappa),$$

$$\zeta_0 = \hat{\zeta}_0(\hat{\mathbf{U}}, \theta, \kappa),$$

respectively. We may, without any difficulty, further restrict the theory to the case when all the response functions are independent of κ.

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REFERENCES

1. P. M. Naghdi, Plasticity. *Proc. 2nd Symp. Naval Structural Mechanics*, 121. Pergamon Press (1960).
2. A. E. Green and P. M. Naghdi, *Arch. ration. Mech. Analysis.* **18**, 251 (1965).
3. A. E. Green and P. M. Naghdi, *Proc. IUTAM Symp. on Irreversible Aspects of Continuum Mechanics.* 117 (1966). Springer (1968).
4. P. Perzyna and W. Wojno, *Archwm. Mech. Stosow.* **20**, 499 (1968).
5. B. D. Coleman and M. E. Gurtin, *J. Chem. Phys.* **47**, 597 (1967).
6. K. Kratochvil and O. W. Dillon, Jr. *J. appl. Phys.* **40**, 3207 (1969).
7. J. Kratochvil and O. W. Dillon Jr. *J. appl. Phys.* **41**, 1470 (1970).
8. R. deWit, *Proc. IUTAM Symp. on the Generalized Cosserat Continuum and the Continuum Theory of Dislocations with Applications* 251 (1967). Springer (1968).
9. E. H. Lee, *J. appl. Mech.* **36**, 1 (1969).
10. L. B. Freund, *Int. J. Solids Struct.* **6**, 1193 (1970).
11. K. M. Kelly, *Archwm. Mech. Stosow.* **22**, 93 (1970).
12. A. E. Green, and P. M. Naghdi, *Mathematika* **12**, 21 (1965).
13. A. E. Green and R. S. Rivlin, *Arch. Ration. Mech. Analysis* **17**, 113 (1964).
14. N. Fox, *Q. J. Mech. appl. Math.* **21**, 67 (1968).
15. A. C. Pipkin and R. S. Rivlin, *Z. Angew. Math. Phys.* **16**, 313 (1965).
16. D. R. Owen, *Arch ration. Mech. Analysis.* **31**, 91 (1968).
17. D. R. Owen, *Arch. ration. Mech. Analysis*, 85 (1970).
18. H. T. Hahn and W. Jaunzemis, A dislocation theory of plasticity. *9th Annual Meeting of the Society of Engineering Science* (1971).
19. M. E. Gurtin and W. O. Williams, *Arch. ration. Mech. Analysis* **23**, 163 (1966).
20. C. C. Wang, *Arch. ration. Mech. Analysis* **33**, 249 (1969); **33**, 268 (1969); **36**, 166 (1970); **36** 198 (1970).
21. A. Phillips, *Proc. IUTAM Symp. on Thermoinelasticity* 241 (1968). Springer (1970).

Абстракт — На основе многократного разложения местной деформации в упругую и пластическую деформации, предполагаются общие уравнения состояния для упруго-пластических материалов. Обсуждаются два вариантные подходы к решению: первый, в котором упругая деформация использована в смысле независимой переменной и второй, котором напряжение является одной из независимых переменных. Определяются свойственные симметрии материала. Доказано, что пластический спин отсутствует в теории изотропных материалов. В качестве примера дается анализ простого растяжения.